less Reynolds number dependent. In view of this reasoning, it is not surprising that for transitional separation at transonic conditions, the transition location can vary around the circumference of a symmetric configuration, thereby producing an asymmetric flow. Within the resulting asymmetric separated region, rather large vortices are created that transport fluid from one region to another around the circumference. When a symmetric configuration is brought to an angle of attack, crossflow introduces nonsymmetric flow separation and vortex motion problably results regardless of the type of separation or the freestream Mach number.

Similar large-scale vortices were observed by Whitehead[8] at Mach 6 within an irregularly shaped separated region on a delta wing at zero angle of attack with a 40° trailing edge flap. As discussed in Ref. 8 the shape of the separated region and the nature of the subsequent vortex formation is similarly dependent on the location of transition across the span. Further confirming this conclusion is the change in the character of the separated region when the local Reynolds number is increased; a more regular separation appears in which the vortices are considerably reduced in size and are confined to the outboard region of the hinge line. The large-scale vortices observed on Reding's axisymmetric model and on the delta wing of Ref. 8 are responsible for transporting a portion of the flow out of the separated region. Thus the classical Chapman-Korst separated-flow model, which assumes that fluid can depart the separated region only by passing through the mixing region adjacent to the dividing streamline, is not applicable in these cases.

In contrast to these large-scale vortices are the small-scale vortices observed by Roshko and Thomke[2] in the reattachment region on an axisymmetric rearward-facing step, and by Ginoux[1,9] on a two-dimensional rearward facing step. The effect of these vortices on the reattachment flow and on the mass transferred out of the separated region has not been fully determined, but detailed pressure distributions in the turbulent reattachment region by Roshko and Thomke[2] indicate only a small effect on the static pressure distribution (less than 2% circumferential variation). Furthermore, while Roshko and Thomke's oil-flow results give evidence of the small-scale vortices in the reattachment region, there is no evidence of the circumferential communication which was observed by Reding et al.[3] Ginoux[9] states that the source of these vortices arises from small irregularities in the leading edge. In his studies, he observed regular spanwise variations in pitot measurements within the reattached boundary layer. The magnitude of the resultant spanwise heating variations was strongly dependent on the type of separation; only a 7% variation was recorded for turbulent reattachment compared to over a 100% variation at reattachment of a transitional-separated flow. In a two-dimensional, turbulent-separation study by Sterrett et al.[10] at Mach 6, oil-flow traces of vortices shed from spherical roughness elements appeared in the vicinity and downstream of reattachment on a 40° wedge. Only small spanwise variations were observed in the heating caused by these vortices.

The assertion by Reding et al.[3] that the presence of vortices in separated flows precludes the existence of symmetric flows does not consider the origin or size of the vortices. The small-scale vortices can be considered small perturbations on an existing flowfield and, as such, have a smaller effect on the flowfield and on surface properties than the large-scale vortices that can alter the flowfield significantly. Transitional separation apparently enhances the formation of large-scale vortices, and maximizes the amplitude of the spanwise variations resulting from any small-scale vortices. The present evidence has shown, however, that a symmetric flow field can exist over symmetric configurations for both transitional and turbulent separation.

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Stability of Dynamic Systems Subjected to Nonconservative and Harmonic Forces

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1. Introduction

CONSIDER a dynamic system represented by a set of linear ordinary differential equations with periodic coefficients as follows:

$$\ddot{x}_\alpha + \Omega_{\alpha\beta}x_\beta + \epsilon\varphi_{\alpha\beta}(t)x_\beta = 0, \quad \alpha,\beta = 1, 2, \ldots, N \quad (1)$$

where $\Omega_{\alpha\beta} = 0$ if $\alpha \neq \beta$ and $\Omega_{\alpha\beta} > 0$ if $\alpha = \beta$, ϵ is an amplitude parameter, $\varphi_{\alpha\beta}$ are symmetric periodic functions of a period T with zero mean values, and where the summation convention on repeated indices is used and will also be employed in the sequel. The stability of this system under a set of rather weak assumptions regarding the periodic functions $\varphi_{\alpha\beta}$ has been extensively studied by a number of investigators, see Cesari,[1] Mettler,[2] Hale,[3–5] Gambill,[5–7] and Cesari and Bailey[8]; for further references see Cesari.[1] These investigations are based on a method originally proposed by Cesari where, by means of a convergent successive approximation technique, explicit relations are obtained for the so-called characteristic exponents of the system.

When a dynamic system is subjected to circulatory, gyroscopic, and other nonconservative forces, Eq. (1) must be

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replaced by

$$\ddot{x}_\alpha + D_{\alpha\beta}\dot{x}_\beta + [A_{\alpha\beta} + \epsilon\psi_{\alpha\beta}(t)]x_\beta = 0 \qquad (1')$$

where $\mathbf{D} = [D_{\alpha\beta}]$ and $\mathbf{A} = [A_{\alpha\beta}]$ are general nonsymmetric matrices, and $\psi = [\psi_{\alpha\beta}]$ is a period, (possibly) nonsymmetric matrix. In this case, the results mentioned above cannot be directly applied to Eq. (1′) without further extension. To remedy this, Bolotin,[9] for the special case in which $\psi_{\alpha\beta}(t)$ are harmonic functions of t, has proposed another approximate method which, while being quite easy to apply, is unfortunately based on an incorrect assumption, and hence may lead to serious errors.‡

Bolotin considers a system of equations which in the present notation may be expressed as

$$\ddot{x}_\alpha + D_{\alpha\beta}\dot{x}_\beta + [A_{\alpha\beta} + \epsilon P_{\alpha\beta}\cos\theta t]x_\beta = 0, \qquad \alpha,\beta = 1, 2, \ldots, N \qquad (2)$$

where $\mathbf{P} = [P_{\alpha\beta}]$ is a general nonsymmetric matrix whose elements may depend on the frequency θ. He then assumes that in the θ, ϵ-plane, similarly to the case of a system with one degree of freedom, the corresponding stable and unstable regions are separated by solutions which are periodic. The fact that this assumption is incorrect even when the matrices $\mathbf{A}$ and $\mathbf{P}$ are symmetric can be established from the work of Mettler[10] and Hsu,[11,12] where it is shown that such systems may admit unstable solutions by the so-called "combination resonance" which cannot be detected by the above scheme.

Notwithstanding its insufficiency, Bolotin's method does have certain computational simplicity and therefore, it appears desirable to see whether it cannot be corrected. And this is the purpose of the present note. Here we shall establish sufficient conditions for the stability of the dynamic system Eq. (2) when ϵ is sufficiently small so that the terms of the order ϵ^3 are negligible. Since we shall place no restriction on the symmetry property of the matrices $\mathbf{D}$, $\mathbf{A}$, and $\mathbf{P}$, in contrast to the previous investigations, our results are directly applicable to linear dynamic systems which include gyroscopic, dissipative, and circulatory forces that may be stationary as well as harmonic. For the sake of simplicity in the presentation, we shall assume $N = 2$. The extension of the results to systems with larger degrees of freedom can be accomplished in a routine manner as will be indicated.

2. Stability Conditions

Let $N = 2$ in Eq. (2). From the theory of linear differential equations with periodic coefficients,[1] it is known that Eq. (2) admits solutions of the form

$$x_\alpha = \varphi_\alpha(t)e^{ht} \qquad (3)$$

where $\varphi_\alpha(t)$ are periodic. We express $\varphi_\alpha(t)$ in a Fourier series as

$$\varphi_\alpha(t) = \sum_{i=1,3,5,\ldots}^{\infty}\left[a_\alpha{}^i \sin\frac{i\theta t}{2} + b_\alpha{}^i \cos\frac{i\theta t}{2}\right] \qquad (4)$$

which corresponds to a case where $\varphi_\alpha(t)$ has a period equal to $4\pi/\theta$; this pertains to the principal region of instability. Now substitution from Eq. (3) and Eq. (4) into Eq. (2) results in the following system of linear equations for the coefficients $a_\alpha{}^i$ and $b_\alpha{}^i$:

$$\{(h^2 - \theta^2/4)\delta_{\alpha\beta} + hD_{\alpha\beta} + [A_{\alpha\beta} - (\epsilon/2)P_{\alpha\beta}]\}a_\beta{}^1 + (\epsilon/2)P_{\alpha\beta}a_\beta{}^3 - [h\theta\delta_{\alpha\beta} + (\theta/2)D_{\alpha\beta}]b_\beta{}^1 = 0$$

$$[(h^2 - \theta^2/4)\delta_{\alpha\beta} + hD_{\alpha\beta} + (A_{\alpha\beta} + (\epsilon/2)P_{\alpha\beta})]b_\beta{}^1 + (\epsilon/2)P_{\alpha\beta}b_\beta{}^3 + [h\theta\delta_{\alpha\beta} + (\theta/2)D_{\alpha\beta}]a_\beta{}^1 = 0$$

$$[(h^2 - i^2\theta^2/2)\delta_{\alpha\beta} + hD_{\alpha\beta} + A_{\alpha\beta}]a_\beta{}^i + (\epsilon/2)P_{\alpha\beta}(a_\beta{}^{i-2} + a_\beta{}^{i+2}) - [ih\theta\delta_{\alpha\beta} + (i\theta/2)D_{\alpha\beta}]b_\beta{}^i = 0$$

$$[(h^2 - i^2\theta^2/2)\delta_{\alpha\beta} + hD_{\alpha\beta} + A_{\alpha\beta}]b_\beta{}^i + (\epsilon/2)P_{\alpha\beta}(b_\beta{}^{i-2} + b_\beta{}^{i+2}) + [ih\theta\delta_{\alpha\beta} + (i\theta/2)D_{\alpha\beta}]a_\beta{}^i = 0 \qquad (5)$$

$$\alpha,\beta = 1,2,\ i = 3,5,7, \ldots$$

where $\mathbf{I} = [\delta_{\alpha\beta}]$ is the identity matrix. This system of homogeneous linear equations admits nontrivial solutions if and only if the determinant of the coefficients of the unknowns, $a_\alpha{}^i$ and $b_\alpha{}^i$, vanishes identically, i.e.,

$$\begin{vmatrix} \ddots & & & & \\ & \left(h^2 - \frac{9\theta^2}{4}\right)\mathbf{I} + h\mathbf{D} + \mathbf{A} & \frac{\epsilon}{2}\mathbf{P} & 0 & -3\left(h\theta\mathbf{I} + \frac{\theta}{2}\mathbf{D}\right) \\ & \frac{\epsilon}{2}\mathbf{P} & \left(h^2 - \frac{\theta^2}{4}\right)\mathbf{I} + h\mathbf{D} + \left(\mathbf{A} - \frac{\epsilon}{2}\mathbf{P}\right) & -\left(h\theta\mathbf{I} + \frac{\theta}{2}\mathbf{D}\right) & 0 \\ & 0 & \left(h\theta\mathbf{I} + \frac{\theta}{2}\mathbf{D}\right) & \left(h^2 - \frac{\theta^2}{4}\right)\mathbf{I} + h\mathbf{D} + \left(\mathbf{A} + \frac{\epsilon}{2}\mathbf{P}\right) & \frac{\epsilon}{2}\mathbf{P} \\ & 3\left(h\theta\mathbf{I} + \frac{\theta}{2}\mathbf{D}\right) & 0 & \frac{\epsilon}{2}\mathbf{P} & \left(h^2 - \frac{9\theta^2}{4}\right)\mathbf{I} + h\mathbf{D} + \mathbf{A} \\ & & & & & \ddots \end{vmatrix} = 0 \qquad (6)$$

It can readily be seen that Eq. (6) may be replaced by the following determinant if the terms of the order ϵ^3 and higher are to be neglected

$$\begin{vmatrix} \left(h^2 - \frac{\theta^2}{4}\right)\mathbf{I} + h\mathbf{D} + \left(\mathbf{A} - \frac{\epsilon}{2}\mathbf{P}\right) & -\left(h\theta\mathbf{I} + \frac{\theta}{2}\mathbf{D}\right) \\ \left(h\theta\mathbf{I} + \frac{\theta}{2}\mathbf{D}\right) & \left(h^2 - \frac{\theta^2}{4}\right)\mathbf{I} + h\mathbf{D} + \left(\mathbf{A} + \frac{\epsilon}{2}\mathbf{P}\right) \end{vmatrix} = 0$$

Now, expanding this determinant, we obtain

$$h^8 + r_1h^7 + \ldots + r_7h + r_8 = 0 \qquad (7)$$

‡ The English translation of Bolotin's book is widely used in English speaking countries by (unfortunately) those who may not be aware of its inadequacies.

where the coefficients r_i, $i = 1, 2, \ldots, 8$ are given by

$$r_1 = 2\lambda,\ r_2 = 2\omega + \lambda^2 + 2D - \theta^2$$

$$r_3 = \lambda(2\omega - \theta^2 + D),\ r_4 = \mu_1 + \mu_2 + 2\lambda\nu + \theta^4 + \xi_1\xi_2$$

$$r_5 = \lambda(\mu_1 + \mu_2) + \xi_1\pi_2 + \xi_2\pi_1$$

$$r_6 = \xi_2\mu_1 + \xi_1\mu_2 + \theta^3 D,\ r_7 = \pi_2\mu_1 + \pi_1\mu_2$$

$$r_8 = \begin{vmatrix} \left(\mathbf{A} - \frac{\epsilon}{2}\mathbf{P} - \frac{\theta^2}{4}\mathbf{I}\right) & -\frac{\theta}{2}\mathbf{D} \\ \frac{\theta}{2}\mathbf{D} & \left(\mathbf{A} + \frac{\epsilon}{2}\mathbf{P} - \frac{\theta^2}{4}\mathbf{I}\right) \end{vmatrix} \tag{8}$$

where $D = |\mathbf{D}|$ is the determinant of the matrix $\mathbf{D}$, and the other terms are defined as follows:

$$\lambda = D_{11} + D_{22},\ \omega = A_{11} + A_{22},\ \mu_{1,2} = \left|\mathbf{A} \pm \frac{\epsilon}{2}\mathbf{P} - \frac{\theta^2}{4}\mathbf{I}\right|$$

$$\xi_{1,2} = \left(A_{11} \pm \frac{\epsilon}{2}P_{11} - \frac{\theta^2}{4}\right) + \left(A_{22} \pm \frac{\epsilon}{2}P_{22} - \frac{\theta^2}{4}\right) + D$$

$$\pi_{1,2} = D_{11}\left(A_{22} \pm \frac{\epsilon}{2}P_{22} - \frac{\theta^2}{4}\right) + D_{22}\left(A_{11} \pm \frac{\epsilon}{2}P_{11} - \frac{\theta^2}{4}\right) - D_{12}\left(A_{21} \pm \frac{\epsilon}{2}P_{21}\right) - D_{21}\left(A_{12} \pm \frac{\epsilon}{2}P_{12}\right)$$

$$\nu = D_{11}\left(A_{22} - \frac{\theta^2}{4}\right) + D_{22}\left(A_{11} - \frac{\theta^2}{4}\right) - (D_{12}A_{21} + D_{21}A_{12}) \tag{9}$$

To ensure the stability of system Eq. (2), we now seek conditions under which Eq. (7) admits only roots with negative real parts. As is well-known,[1] these conditions are given by the Routh-Hurwitz criterion which for the present case becomes

$$D_i = \begin{vmatrix} r_1 & 1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ r_3 & r_2 & r_1 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ r_5 & r_4 & r_3 & r_2 & r_1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ r_{2i-1} & r_{2i-2} & r_{2i-3} & \cdot & \cdot & \cdot & \cdot & \cdot & r_i \end{vmatrix} > 0 \tag{10}$$

$$i = 1, 2, \ldots, 8$$

where in each determinant, all r's having negative subscripts or having subscripts larger than 8 are to be set equal to zero. We note that, since $D_8 = r_8 D_7$, the last inequality in Eq. (10) can be replaced by

$$r_8 > 0 \tag{11}$$

From the results presented above, it must be clear that the requirement (11) alone is not sufficient to guarantee stability whether the matrices $\mathbf{D}$, $\mathbf{A}$, and $\mathbf{P}$ are symmetric or not; even when these matrices are symmetric, the possibility of combination resonance is not ruled out by Eq. (11) alone. On the other hand, when $\mathbf{A}$ is a nonsymmetric matrix, the corresponding autonomous system [obtained by setting $\epsilon = 0$ in Eq. (2)],

$$\ddot{x}_\alpha + D_{\alpha\beta}\dot{x}_\beta + A_{\alpha\beta}x_\beta = 0,\ \alpha,\beta = 1,2 \tag{12}$$

may lose stability by flutter-type motions.[13] The requirement (11) with $\epsilon = 0$, however, is the usual static stability condition which, as has been shown by Ziegler,[14] is inadequate for flutter analysis; hence, all the conditions corresponding to Eq. (10) must be considered. The inequality (11) is the sole condition required by Bolotin[9] for ensuring stability.§ We see that (11) is not only inadequate to prevent combination resonance, but also it provides no guarantee against other types of dynamic instability modes which are peculiar to circulatory systems, especially when harmonic forces are also present.

For a more general system with N degrees of freedom, Eq. (7) changes to a polynomial of the degree $4N$ in h. The corresponding stability conditions can be readily constructed from Eq. (10). When the damping matrix $\mathbf{D}$ is zero, conditions (10) must be slightly modified, but we shall not report these results here.

In summary, we point out that conditions (10) are sufficient to guarantee the stability of a general linear dynamic system subjected to nonconservative forces with harmonic components, provided that the latter forces are sufficiently small. These conditions include Bolotin's stability criterion which by itself is inadequate and may lead to serious errors. The method is not readily applicable to systems of equation with general periodic coefficients even when these periodic coefficients admit Fourier series expansions. In this case, one must seek an extension of Cesari's method or the method proposed by Hsu[11] for application to systems of type (1′).

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§ The English translation of this book contains a short section (Sec. 60) on combination resonance. This section is then terminated by stating that "a detailed investigation is necessary, however, in order to verify completely the practical significance of the combination resonance."